Lineare partielle Differentialgleichungen 1. Ordnung

Eine PDGL mit übersichtlicher, aber nicht trivialer Lösung

In[◦]:=
```
solu = DSolve[x ∂x u[x, y] + y ∂y u[x, y] == 2 u[x, y] + 1, u[x, y], {x, y}]
```
Out[◦]=

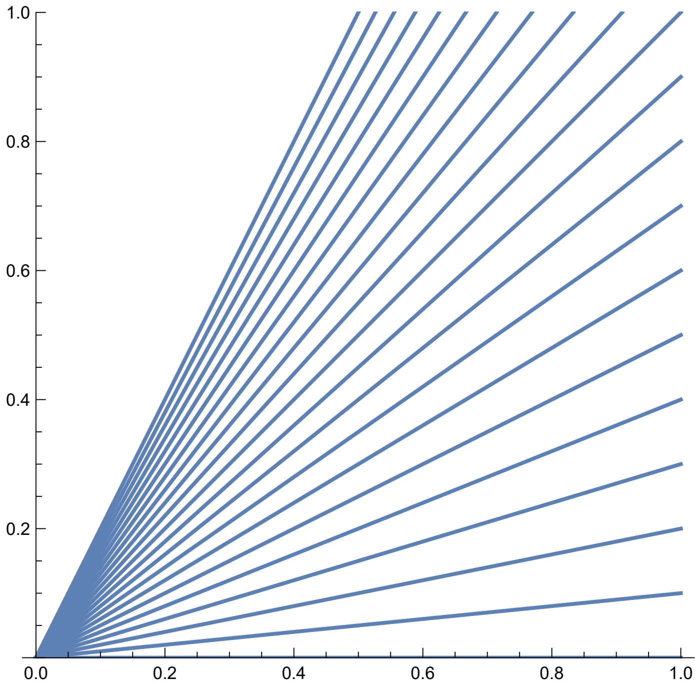
$$\left\{\left\{u[x, y] \to -\frac{1}{2} + x^2 \, c_1\left[\frac{y}{x}\right]\right\}\right\}$$

Die Projektionen der charakteristischen Kurven auf die xy-Ebene sehen so aus:

In[◦]:=
```
solyx = DSolve[y'[x] == y[x]/x, y[x], x]
```
Out[◦]=

$$\{\{y[x] \to x \, c_1\}\}$$

In[◦]:=
```
projchar = Plot[Table[y[x] /. solyx[[1]] /. C[1] → c, {c, 0, 2, 0.1}],
   {x, 0, 1}, PlotRange → {0, 1}, AspectRatio → Automatic]
```
Out[◦]=



In[◦]:=
```
sol1 = DSolve[{x'[t] == x[t], y'[t] == y[t], z'[t] == 2 z[t] + 1},
    {x[t], y[t], z[t]}, t] // Simplify
```
Out[◦]=

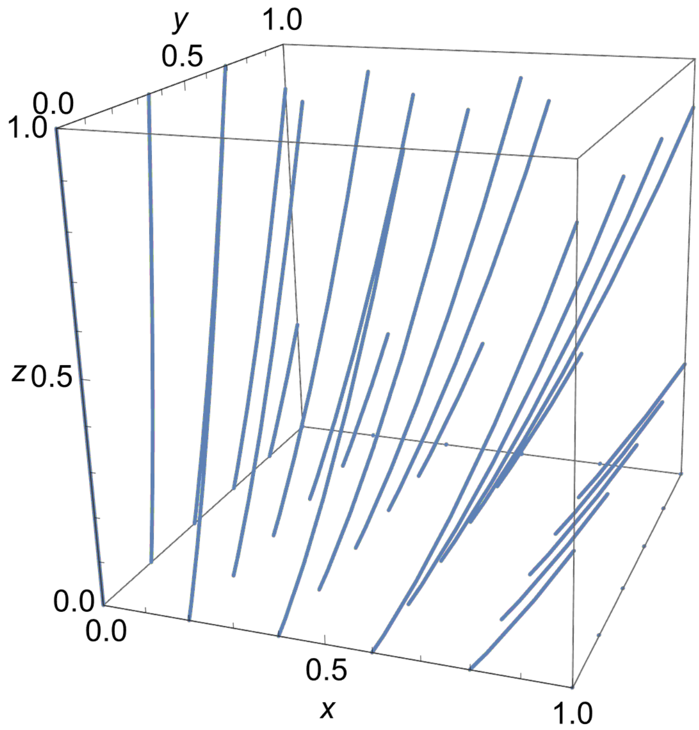
$$\left\{\left\{x[t] \to e^t \, c_1,\ y[t] \to e^t \, c_2,\ z[t] \to -\frac{1}{2} + e^{2t} \, c_3\right\}\right\}$$

Hier kommen die charak. Kurven in voller Gestalt:

In[◦]:=
```
curves = Flatten[Table[{x[t], y[t], z[t]} /. sol1 /. {C[1] → c1, C[2] → c2, C[3] → 0.5},
     {c1, -2, 1, 0.2}, {c2, 0, 1, 0.2}], 2];
```

In[∘]:=
```
chars = ParametricPlot3D[curves, {t, -2, 3},
  PlotRange → {{0, 1}, {0, 1}, {0, 1}}, AxesLabel → {x, y, z}, BaseStyle → FontSize → 16]
```

Out[∘]=



Eine nicht ganz triviale Lösungsfläche

In[∘]:=
$$F = \left(\frac{1}{2\,\#^2}\ \&\right);$$

In[∘]:=
```
uterm = u[x, y] /. solu /. C[1] → F // Simplify
```

Out[∘]=
$$\left\{\frac{1}{2}\left(-1+\frac{x^4}{y^2}\right)\right\}$$

In[∘]:=
```
Solve[uterm == 0]
```

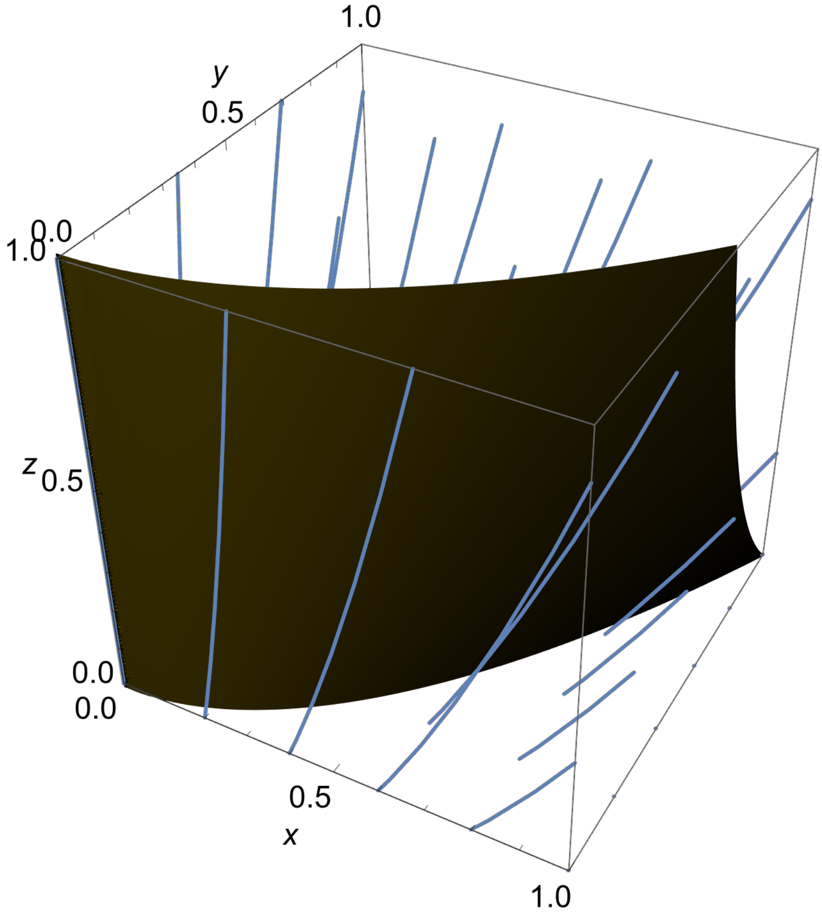
Out[∘]=
$$\{\{y \to -x^2\}, \{y \to x^2\}\}$$

In[∘]:=
```
surface0 = Plot3D[{uterm}, {x, 0, 1}, {y, 0, 1}, PlotPoints → 300,
  Mesh → None, Lighting → DirectionalLight[RGBColor[0.2, 0.2, 0], {0.5, 0, 1}]]
```

... und mit den charak. Kurven.

In[•]:= `Show[chars, surface0, BoxRatios → Automatic]`

Out[•]=



Hier trifft diese Fläche die xy-Ebene: das ist dann KEINE projizierte charak. Kurve!

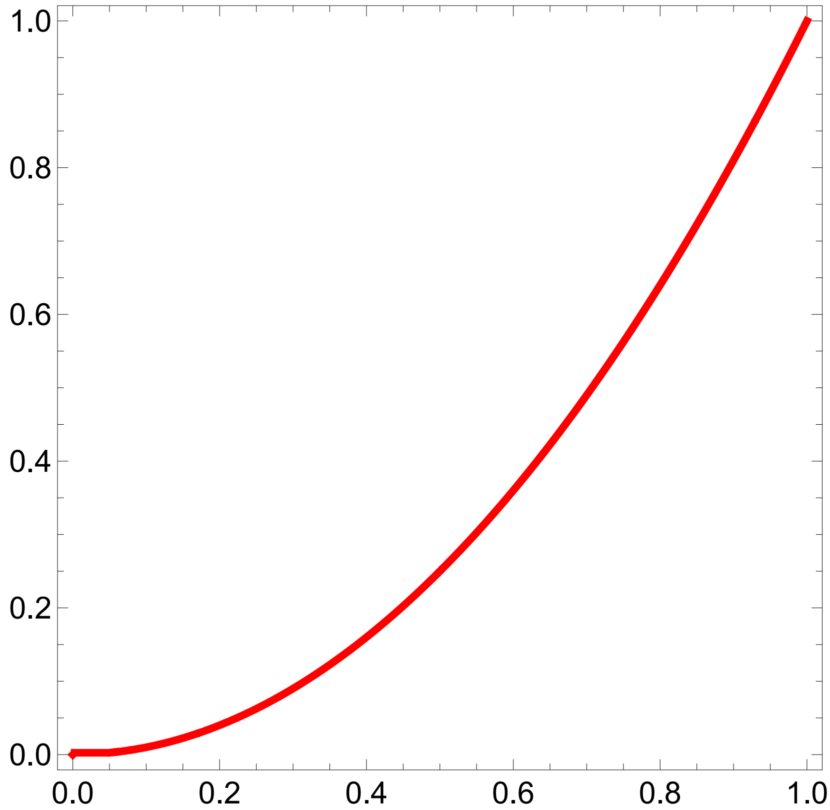
In[•]:= `impleq = uterm == 0`

Out[•]=

$$\left\{\frac{1}{2}\left(-1+\frac{x^4}{y^2}\right)\right\} == 0$$

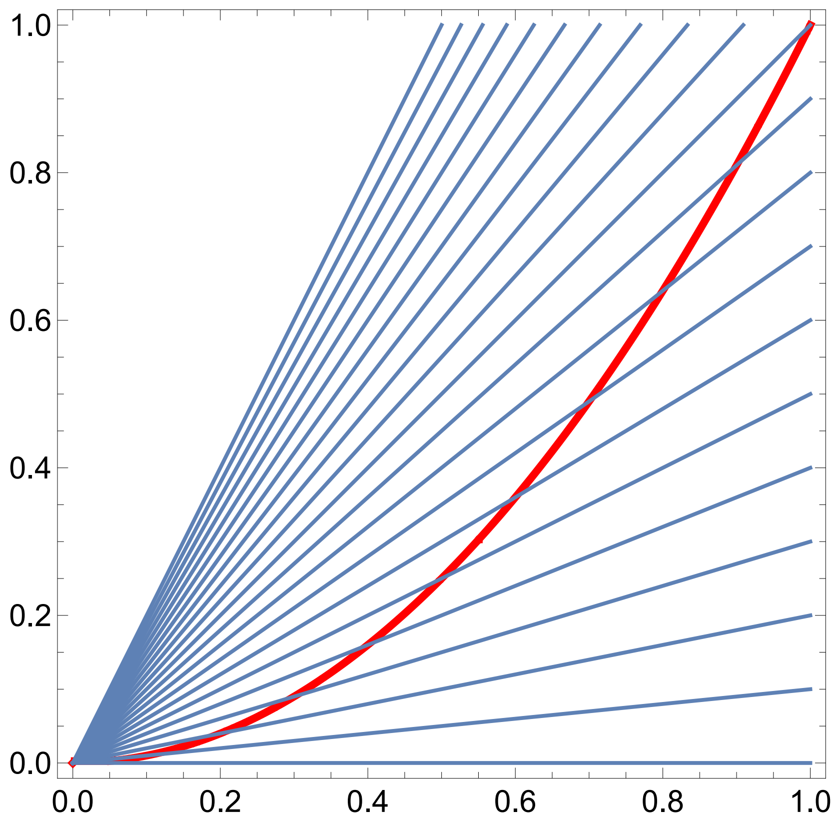
In[•]:=
```
xycut = ContourPlot[impleq // Evaluate, {x, 0, 1}, {y, 0, 1},
  ContourStyle → {Thickness[0.01], Red}, PlotPoints → 100, BaseStyle → FontSize → 16]
```

Out[•]=



In[•]:=
```
Show[xycut, projchar]
```

Out[•]=



In[∘]:=
```
soly = Solve[impleq, y]
```
Out[∘]=
$\{\{y \to -x^2\}, \{y \to x^2\}\}$

In[∘]:=
```
soly /. x → 0.2
```
Out[∘]=
```
{{y → -0.04}, {y → 0.04}}
```

In[∘]:=
```
tabx = Table[x, {x, 0, 1, 0.1}];
```

In[∘]:=
```
taby = y /. soly[[2]] /. x → tabx
```
Out[∘]=
```
{0., 0.01, 0.04, 0.09, 0.16, 0.25, 0.36, 0.49, 0.64, 0.81, 1.}
```

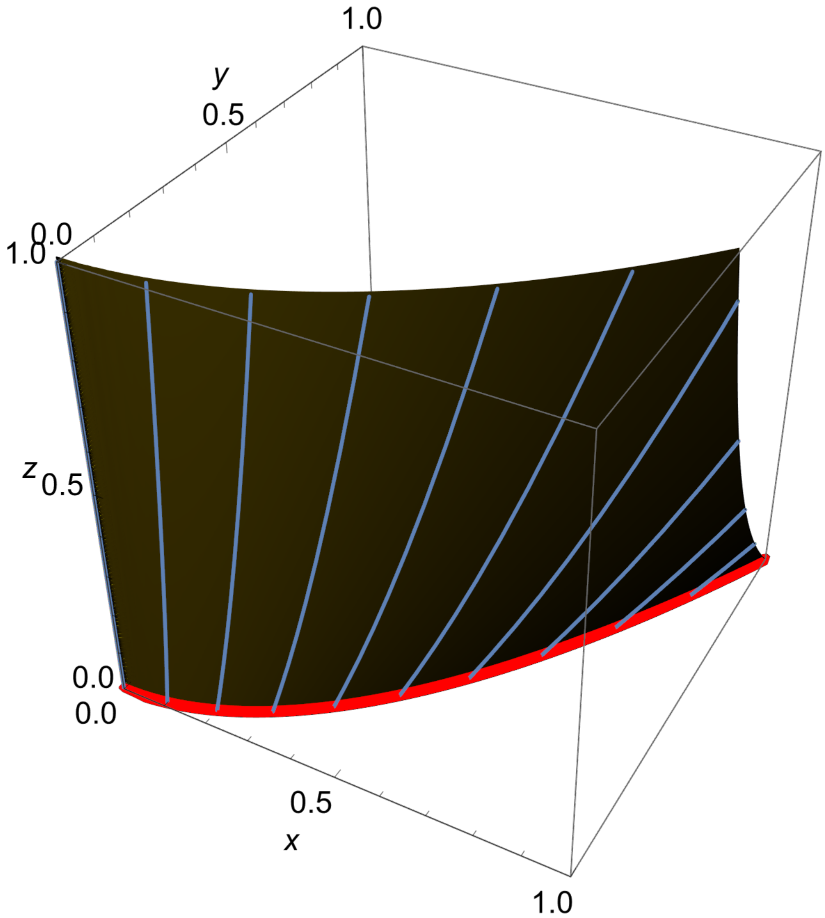
In[∘]:=
```
curvesf = Flatten[Table[{x[t], y[t], z[t]} /. sol1 /.
      {C[1] → tabx[[i]], C[2] → taby[[i]], C[3] → 0.5}, {i, 1, Length@tabx}], 1];
```

Hier werden genau die charak. Kurven gezeigt, die in der Lösungsfläche liegen.

In[∘]:=
```
charsf = ParametricPlot3D[curvesf, {t, 0, 3}, PlotRange → {{0, 1}, {0, 1}, {0, 1}},
    AxesLabel → {x, y, z}, BaseStyle → FontSize → 16];
```

In[∘]:=
```
xycut3D = ParametricPlot3D[{t, t^2, 0}, {t, 0, 1}, PlotRange → {{0, 1}, {0, 1}, {0, 1}},
    PlotStyle → {Thickness[0.015], Red}, PlotPoints → 100];
```

In[∘]:=
```
Show[charsf, surface0, xycut3D, BoxRatios → Automatic]
```
Out[∘]=



Eine zweite Lösungsfläche

In[∘]:=
```
s^-2/2 /. Solve[(s - 1/4)/s == y, s][[1]] /. y → s
```
Out[∘]=

$8\ (-1 + s)^2$

In[∘]:=
```
F = (8 (1 - #) ^2 &);
```

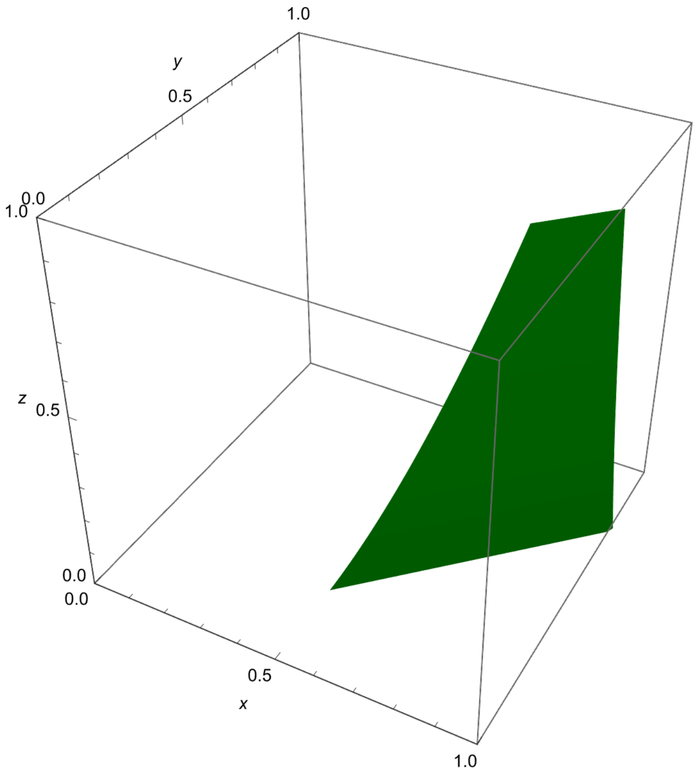
In[∘]:=
```
uterm2 = u[x, y] /. solu /. C[1] → F // Simplify
```
Out[∘]=

$\left\{-\frac{1}{2} + 8\,x^2 - 16\,x\,y + 8\,y^2\right\}$

In[∘]:=
```
Solve[uterm2 == 0]
```
Out[∘]=

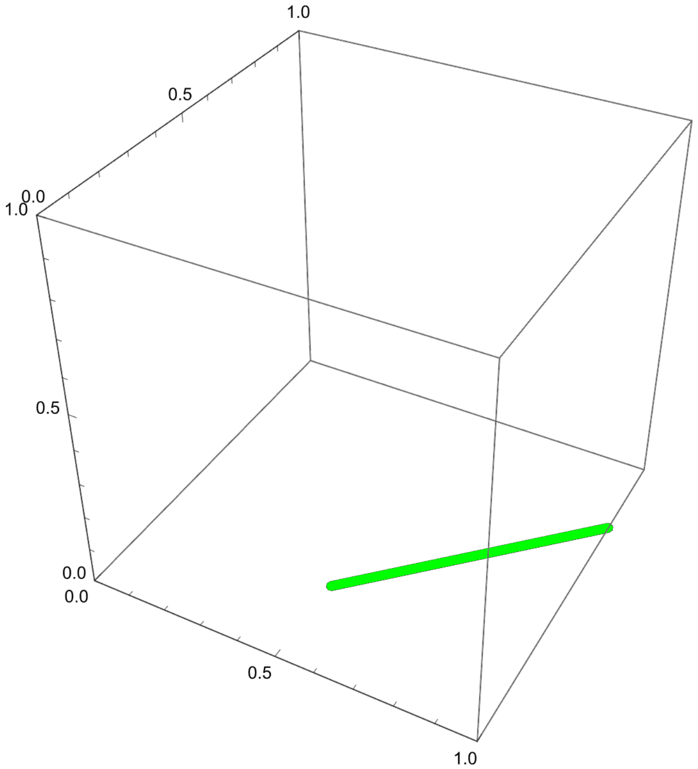
$\left\{\left\{y \to -\frac{1}{4} + x\right\}, \left\{y \to \frac{1}{4} + x\right\}\right\}$

In[∘]:=
```
surface2 = ParametricPlot3D[{t x, (x - 1/4 - 0.005) t, 1/2 t^2 - 1/2}, {x, 0.5, 1},
  {t, 1, 2}, PlotPoints → 300, Mesh → None, PlotRange → {{0, 1}, {0, 1}, {0, 1}},
  AxesLabel → {x, y, z}, Lighting → DirectionalLight[RGBColor[0, 0.5, 0], {1, 0, 1}],
  ColorFunction → Function[{x, y, z}, White]]
```
Out[∘]=



In[ ]:=
```
xycutneu3D =
 ParametricPlot3D[{t, -1/4 + t, 0}, {t, 0.5, 1}, PlotRange → {{0, 1}, {0, 1}, {0, 1}},
  PlotStyle → {Thickness[0.015], Green}, PlotPoints → 100]
```

Out[ ]=



die charak. Kurven gezeigt, die in der zweiten Lösungsfläche liegen

In[ ]:=
```
tabx2 = Table[x, {x, 0.5, 1, 0.1}]
```

Out[ ]=
```
{0.5, 0.6, 0.7, 0.8, 0.9, 1.}
```

In[ ]:=
```
taby2 = tabx2 - 1/4
```

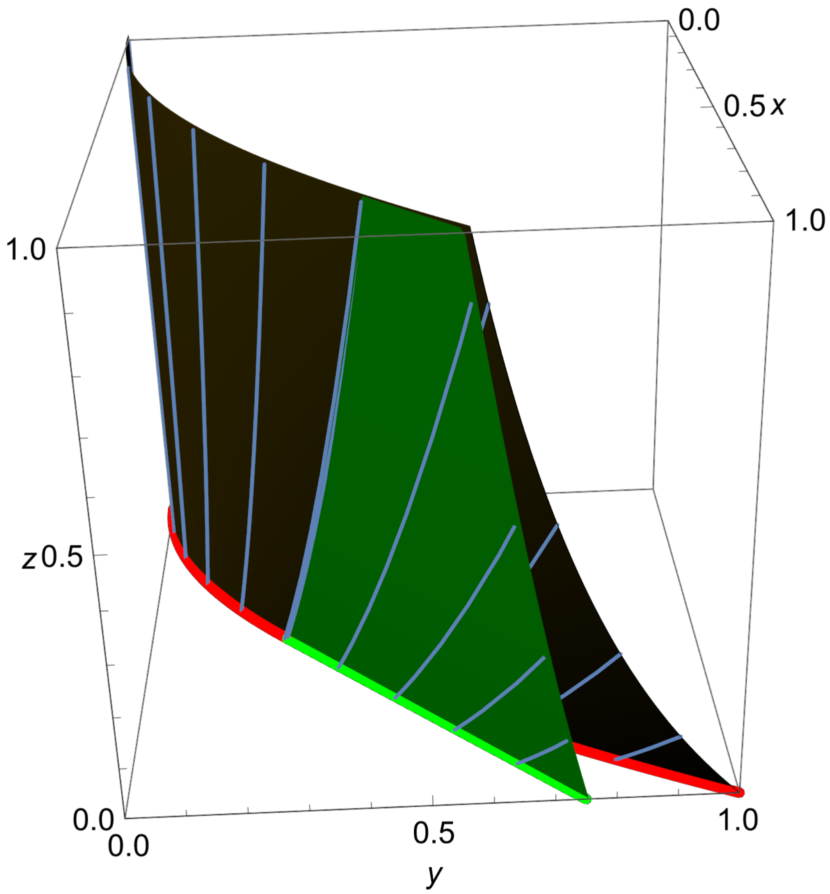
Out[ ]=
```
{0.25, 0.35, 0.45, 0.55, 0.65, 0.75}
```

In[ ]:=
```
curvesf2 = Flatten[Table[{x[t], y[t], z[t]} /. sol1 /.
     {C[1] → tabx2[[i]], C[2] → taby2[[i]] - 0.005, C[3] → 0.5}, {i, 1, Length@tabx2}], 1];
```

In[ ]:=
```
charsf2 = ParametricPlot3D[curvesf2, {t, 0, 4},
   PlotRange → {{0, 1}, {0, 1}, {0, 1}}, AxesLabel → {x, y, z}];
```

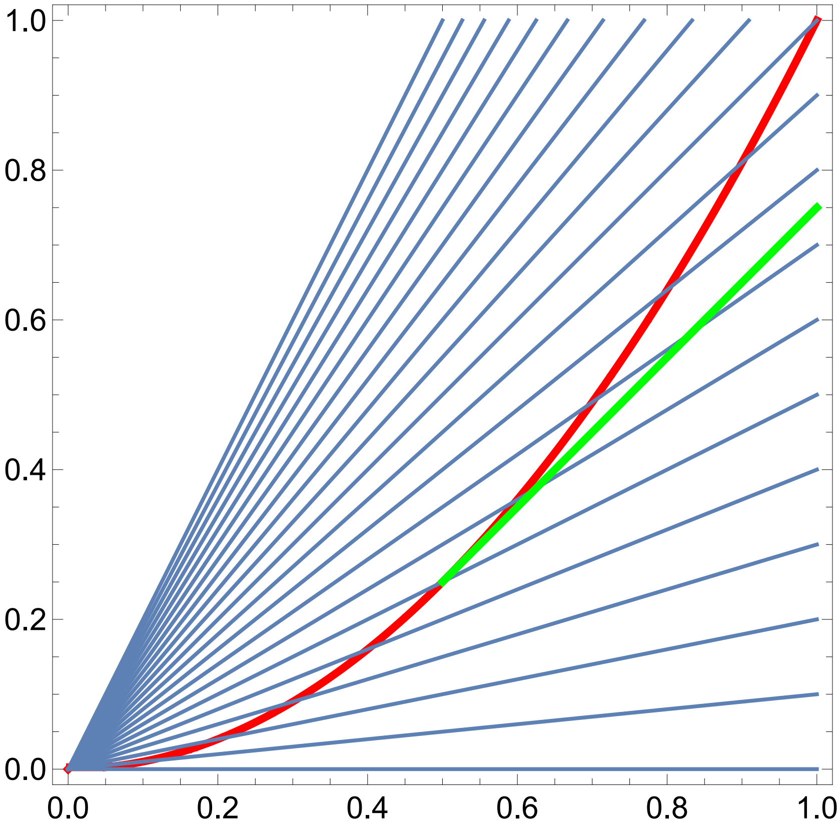
In[ ]:=
```
Show[charsf, charsf2, surface0, surface2, xycut3D, xycutneu3D, AxesLabel → {x, y, z}]
```

Out[ ]=



In[*]:=
```
Show[xycut, projchar, Plot[x - 1/4, {x, 0.5, 1}, PlotStyle → {Thickness[0.01], Green}]]
```

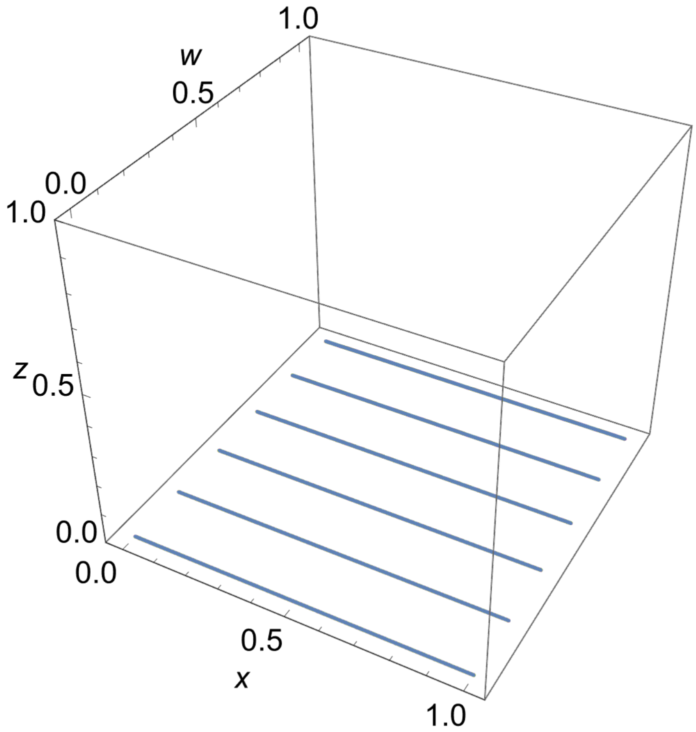
Out[*]=



weitere Bilder zum Text

In[∘]:=
```
projcharw =
 ParametricPlot3D[Table[{x, c, 0}, {c, 0, 1, 0.2}], {x, 0, 1}, PlotRange → {0, 1},
  AspectRatio → Automatic, BaseStyle → FontSize → 16, AxesLabel → {x, w, z}]
```

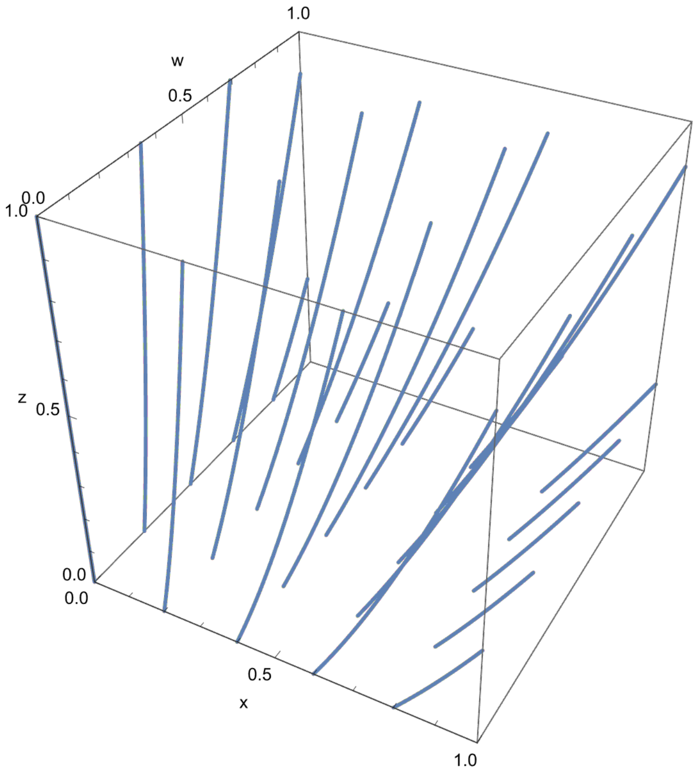
Out[∘]=



In[∘]:=
```
curvesw =
  Flatten[Table[{C[1] e^t, C[2], C[3] e^(2 t) - 1/2} /. {C[1] → c1, C[2] → c2, C[3] → 0.5},
    {c1, -2, 1, 0.2}, {c2, 0, 1, 0.2}], 1];
```

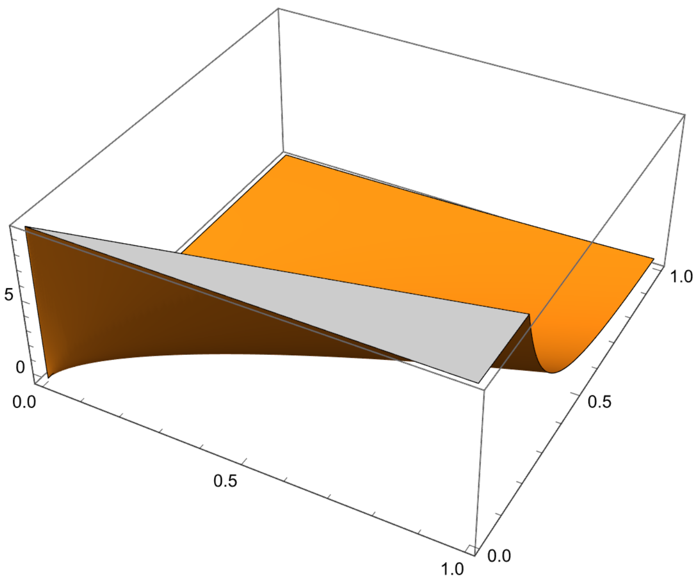
```
In[◦]:= charsw = ParametricPlot3D[curves, {t, 0, 1},
          PlotRange → {{0, 1}, {0, 1}, {0, 1}}, AxesLabel → {"x", "w", "z"}]
```

Out[◦]=



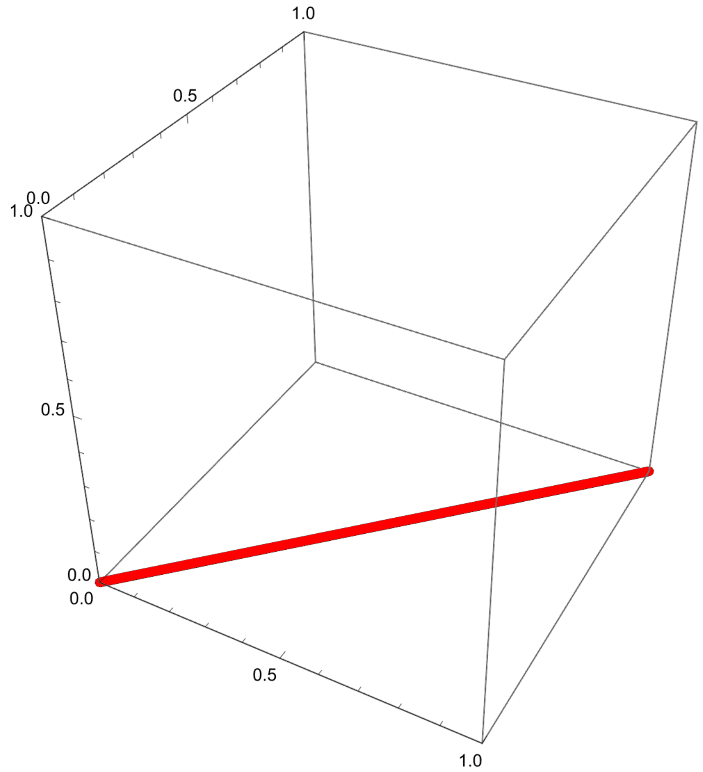
In[◦]:= surfacev = Plot3D[{$\frac{1}{2}\left(\frac{x^2}{w^2}-1\right)$}, {x, 0, 1}, {w, 0, 1}, PlotPoints → 300, Mesh → None]

Out[◦]=



In[ ]:=
```
xwcutneu3D = ParametricPlot3D[{t, t, 0}, {t, 0, 1}, PlotRange → {{0, 1}, {0, 1}, {0, 1}},
  PlotStyle → {Thickness[0.015], Red}, PlotPoints → 100]
```

Out[ ]=



In[ ]:= `Show[projcharw, charsw, surfacev, xwcutneu3D]`

Out[ ]=